Solutions to selected exercises from Jehle and Reny (2001): Advanced Microeconomic Theory

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Contents

1 Mathematical Appendix

1.1 Chapter A1

A1.7 Graph each of the following sets. If the set is convex, give a proof. If it is not convex, give a counterexample.
Answer

(a) $(x, y)|y = e^x$
This set is not convex.
Any combination of points would be outside the set. For example, $(0, 1)$ and $(1, e) \in (x, y)|y = e^x$, but combination of the two vectors with $t = \frac{1}{2}$ not: $(\frac{1}{2}, \frac{e+1}{2}) \notin (x, y)|y = e^x$.

(b) $(x, y)|y \geq e^x$
This set is convex.
Proof: Let (x_1, y_1), $(x_2, y_2) \in S = (x, y)|y \geq e^x$. Since $y = e^x$ is a continuous function, it is sufficient to show that $(tx_1 + (1-t)x_2,\ ty_1 + (1-t)y_2) \in S$ for any particular $t \in (0, 1)$. Set $t = \frac{1}{2}$. Our task is to show that $\left(\frac{1}{2}(x_1 + x_2),\ \frac{1}{2}(y_1 + y_2)\right) \in S$. $\frac{1}{2}(y_1 + y_2) \geq \frac{1}{2}(e^{x_1} + e^{x_2})$, since $y_i \geq e^{x_1}$ for $i = 1, 2$. Also,

$$\frac{1}{2}(e^{x_1} + e^{x_2}) \geq e^{\frac{1}{2}(x_1 + x_2} = e^{\frac{x_1}{2}} \cdot e^{\frac{x_2}{2}}$$
$$\Leftrightarrow e^{x_1} + e^{x_2} \geq 2e^{\frac{x_1}{2}} \cdot e^{\frac{x_2}{2}}$$
$$\Leftrightarrow e^{x_1} - 2e^{\frac{x_1}{2}} \cdot e^{\frac{x_2}{2}} + e^{x_2} \geq 0 \Leftrightarrow (e^{x_1} - e^{x_2})^2 \geq 0.$$

(c) $(x, y)|y \geq 2x - x^2;\ x > 0, y > 0$
This set is not convex.
For example, $\left(\frac{1}{10}, \frac{1}{2}\right), \left(1\frac{9}{10}, \frac{1}{2}\right) \in S = (x, y)|y \geq 2x - x^2; x > 0, y > 0$. However, $\left(1, \frac{1}{2}\right) = \frac{1}{2}\left(\frac{1}{10}, \frac{1}{2}\right) + \frac{1}{2}\left(1\frac{9}{10}, \frac{1}{2}\right) \notin S$

(d) $(x, y)|xy \geq 1;\ x > 0, y > 0$
This set is convex.
Proof: Consider any (x_1, y_1), $(x_2, y_2) \in S = (x, y)|xy \geq 1;\ x > 0, y > 0$. For any $t \in [0, 1]$,

$$\begin{aligned}(tx_1 + (1-t)x_2)(ty_1 + (1-t)y_2) &= t^2x_1y_1 + t(1-t)(x_1y_2 + x_2y_1) + (1-t)^2x_2y_2 \\ &> t^2 + (1-t)^2 + t(1-t)(x_1y_2 + x_2y_1), \text{ since } x_iy_i > 1. \\ &= 1 + 2t^2 - 2t + t(1-t)(x_1y_2 + x_2y_1) \\ &= 1 + 2t(t-1) + t(1-t)(x_1y_2 + x_2y_1) \\ &= 1 + t(1-t)(x_1y_2 + x_2y_1 - 2) \geq 1 \text{ iff } x_1y_2 + x_2y_1 \geq 0.\end{aligned}$$

$$\begin{aligned}x_1y_2 + x_2y_1 = x_1y_1\frac{y_2}{y_1} + x_2y_2\frac{y_1}{y_2} - 2 \geq \frac{y_2}{y_1} + \frac{y_1}{y_2} - 2 \geq 0 \\ y - 1 - 2y_1y_2 + y_2 \geq 0 \\ (y_1 - y_2)^2 \geq 0,\end{aligned}$$

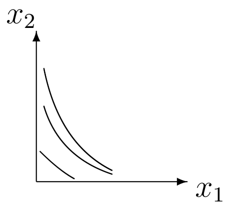
which is always true and therefore, $(tx_1 + (1-t)x_2, ty_1 + (1-t)y_2) \in S$ which is convex.

(e) $(x, y)|y \leq \ln(x)$
This set is convex.
Proof. Let $(x_1, y_1) + (x_2, y_2) \in S$. Then $\frac{1}{2}(y_1 + y_2) \leq (\ln(x_1) + \ln(x_2))$.
S is convex

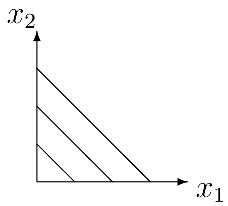
$$\begin{aligned}
&\overset{if}{\Rightarrow} \frac{1}{2}\left(\ln(x_1) + \ln(x_2) \leq \ln(\frac{1}{2}x_1 + \frac{1}{2}x_2)\right) \\
&\Leftrightarrow \frac{1}{2}\ln(x_1 x_2) \leq \ln(\frac{1}{2}x_1 + \frac{1}{2}x_2) \\
&\Leftrightarrow (x_1 x_2)^{1/2} \leq (\frac{1}{2}x_1 + \frac{1}{2}x_2) \\
&\Leftrightarrow x_1 - 2(x_1 x_2)^{1/2} + x_2 \geq 0 \\
&\Leftrightarrow \left(x_1^{1/2} + x_2^{1/2}\right)^2 \geq 0
\end{aligned}$$

which is always true.

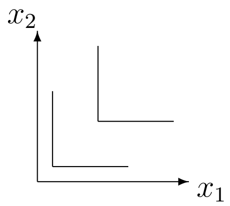
A1.40 Sketch a few level sets for the following functions: $y = x_1 x_2$, $y = x_1 + x_2$ and $y = \min[x_1, x_2]$.
Answer



(a) $y = x_1 x_2$



(b) $y = x_1 + x_2$



(c) $y = \min(x_1, x_2)$

Figure 1: Sets to Exercise A1.40

A1.42 Let $D = [-2, 2]$ and $f : D \to \mathbb{R}$ be $y = 4 - x^2$. Carefully sketch this function. Using the definition of a concave function, prove that f is concave. Demonstrate that the set A is a convex set.
Answer Proof of concavity: Derive the first and second order partial derivative:

$$\frac{\partial y}{\partial x} = -2x \qquad \frac{\partial^2 y}{\partial x^2} = -2$$

The first derivative is strictly positive for values $x < 0$ and negative for values $x > 0$. The second order partial derivative is always less than zero. Therefore, the function is concave.
Proof of convexity: The area below a concave function forms a convex set (Theorem

A1.13). Alternatively, from the definition of convexity the following inequality should hold $4 - (tx^1 + (1-t)x^2)^2 \geq t(4 - (x^1)^2) + (1-t)(4 - (x^2)^2)$. Multiply out to get $4 - (tx^1 + x^2 - tx^2)^2 \geq 4 - x_2^2 + t[(x^1)^2 - (x^2)^2]$. Again, the area below the function forms a convex set.

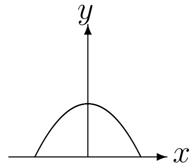


Figure 2: Graph to Exercise A1.42

A1.46 Consider any linear function $f(\mathbf{x}) = \mathbf{a} \cdot \mathbf{x} + b$ for $\mathbf{a} \in \mathbb{R}^n$ and $b \in \mathbb{R}$.

(a) Show that every linear function is both concave and convex, though neither is strictly concave nor strictly convex.
Answer The statement is true iff, for any $\mathbf{x}^1, \mathbf{x}^2 \in \mathbb{R}^n$, $t \in [0,1]$, it is true that

$$f(t\mathbf{x}^1 + (1-t)\mathbf{x}^2) = tf(\mathbf{x}^1) + (1-t)f(\mathbf{x}^2).$$

Substituting any linear equation in this statement gives

$$f(t\mathbf{x}^1 + (1-t)\mathbf{x}^2) = a[t\mathbf{x}^1 + (1-t)\mathbf{x}^2] + b = ta\mathbf{x}^1 + (1-t)a\mathbf{x}^2 + tb + (1-t)b = tf(\mathbf{x}^1) + (1-t)f(\mathbf{x}^2)$$

for all $\mathbf{x}^1, \mathbf{x}^2 \in \mathbb{R}^n$, $t \in [0,1]$.

(b) Show that every linear function is both quasiconcave and quasiconvex and, for $n > 1$, neither strictly so. (There is a slight inaccuracy in the book.)
Answer As it is shown in (a) that a linear function is concave and convex, it must also be quasiconcave and quasiconvex (Theorem A1.19). More formally, the statement is true iff, for any $\mathbf{x}^1, \mathbf{x}^2 \in \mathbb{R}^n \quad (x^1 \neq x^2)$ and $t \in [0,1]$, we have

$$f(t\mathbf{x}^1 + (1-t)\mathbf{x}^2) \geq \min[f(\mathbf{x}^1), f(\mathbf{x}^2)](\text{quasiconcavity})$$
$$f(t\mathbf{x}^1 + (1-t)\mathbf{x}^2) \leq \max[f(\mathbf{x}^1), f(\mathbf{x}^2)](\text{quasiconvexity})$$

Again by substituting the equation into the definition, we get

$$tf(\mathbf{x}^1) + (1-t)f(\mathbf{x}^2) \geq \min[f(\mathbf{x}^1), f(\mathbf{x}^2)]$$
$$tf(\mathbf{x}^1) + (1-t)f(\mathbf{x}^2) \leq \max[f(\mathbf{x}^1), f(\mathbf{x}^2)] \qquad \forall t \in [0,1]$$

A1.47 Let $f(\mathbf{x})$ be a concave (convex) real-valued function. Let $g(t)$ be an increasing concave (convex) function of a single variable. Show that the composite function, $h(\mathbf{x}) = g(f(\mathbf{x}))$ is a concave (convex) function.
Answer The composition with an affine function preserves concavity (convexity). Assume that both functions are twice differentiable. Then the second order partial derivative of the composite function, applying chain rule and product rule, is defined as

$$h''(x) = g''(f(x))\, f'(x)^2 + g'(f(x))\, f''(x)^2$$

For any concave function, $\nabla^2 f(x) \leq 0$, $\nabla^2 g(x) \leq 0$, it should hold $\nabla^2 h(x) \leq 0$. In the case the two functions are convex: $\nabla^2 f(x) \geq 0$ and $\nabla^2 g(x) \geq 0$, it should hold $\nabla^2 h(x) \geq 0$.

A1.48 Let $f(x_1, x_2) = -(x_1 - 5)^2 - (x_2 - 5)^2$. Prove that f is quasiconcave.
Answer Proof: f is concave iff $\mathbf{H}(\mathbf{x})$ is negative semidefinite and it is strictly concave if the Hessian is negative definite.

$$\mathbf{H} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$$
$$\mathbf{z}^T \mathbf{H}(\mathbf{x})\mathbf{z} = -2z_1^2 - 2z_2^2 < 0, \text{ for } \mathbf{z} = (z_1, z_2) \neq 0$$

Alternatively, we can check the leading principal minors of $\mathbf{H}$: $H_1(\mathbf{x}) = -2 < 0$ and $H_2(\mathbf{x}) = 4 > 0$. The determinants of the Hessian alternate in sign beginning with a negative value. Therefore, the function is even strictly concave. Since f is concave, it is also quasiconcave.

A1.49 Answer each of the following questions "yes" or "no", and justify your answer.

(a) Suppose $f(x)$ is an increasing function of one variable. Is $f(x)$ quasiconcave?
Answer Yes, an increasing function of one variable is quasiconcave. Any convex combination of two points on this function will be at least as large as the smallest of the two points. Using the differential-based approach, f is quasiconcave, if for any x^0 and x^1, $f(x^1) \geq f(x^0) \Rightarrow \partial f(x^0)/\partial x(x^1 - x^0) \geq 0$. This must be true for any increasing function.

(b) Suppose $f(x)$ is a decreasing function of one variable. Is $f(x)$ quasiconcave?
Answer Yes, a decreasing function of one variable is quasiconcave. Similarly to (a), f is quasiconcave if for any x^0, x^1 and $t \in [0, 1]$, it is true that $f(tx^0 + (1 - t)x^1) \geq \min[f(x^0), f(x^1)]$.

(c) Suppose $f(x)$ is a function of one variable and there is a real number b such that $f(x)$ is decreasing on the interval $(-\inf, b]$ and increasing on $[b, +\inf)$. Is $f(x)$ quasiconcave?
Answer No, if f is decreasing on $(-\inf, b]$ and increasing on $[b, +\inf)$ then $f(x)$ is not quasiconcave.
Proof: Let $a < b < c$, and let $t_b = \frac{c-b}{c-a} \in [0, 1]$, $t_b a + (1 - t_b)c = b$. Given the nature of f, $f(b) < \min[f(a), f(c)]$. Then $f(t_b a + (1 - t_b)c) < \min[f(a), f(c)]$, so f is not quasiconcave.

(d) Suppose $f(x)$ is a function of one variable and there is a real number b such that $f(x)$ is increasing on the interval $(-\inf, b]$ and decreasing on $[b, +\inf)$. Is $f(x)$ quasiconcave?
Answer Yes.
Proof: Let $a < b < c$, for $x \in [a, b]$, $f(x) \geq f(a)$ and for $x \in [b, c]$, $f(x) \geq f(c)$. Hence, for any $x \in [a, c]$, $f(x) \geq \min[f(a), f(c)]$.

(e) You should now be able to come up with a characterization of quasiconcave functions of one variable involving the words "increasing" and "decreasing".
Answer Any function of one variable $f(x)$ is quasiconcave if and only if is either continuously increasing, continuously decreasing or first increasing and later decreasing.

1.2 Chapter A2

A2.1 Differentiate the following functions. State whether the function is increasing, decreasing, or constant at the point $x = 2$. Classify each as locally concave, convex, or linear at the point $x = 2$.

(a) $f(x) = 11x^3 - 6x + 8 \quad f_1 = 33x^2 - 6$
increasing locally convex

(b) $f(x) = (3x^2 - x)(6x + 1) \quad f_1 = 54x^2 - 6x - 1$
increasing locally convex

(c) $f(x) = x^2 - \dfrac{1}{x^3} \quad f_1 = 2x + \dfrac{3}{x^4}$
increasing locally concave

(d) $f(x) = (x^2 + 2x)^3 \quad f_1 = (6x + 6)(x^2 + 2x)^2$
increasing locally convex

(e) $f(x) = [3x/(x^3 + 1)]^2 \quad f_1 = 18x\dfrac{x^3 - 3x^2 + 1}{(x^3 + 1)^3}$
increasing locally concave

(f) $f(x) = [(1/x^2 + 2) - (1/x - 2)]^4 \quad f_1 = \left(\dfrac{4}{x^2} - \dfrac{8}{x^3}\right)\left(\dfrac{1}{x^2} - \dfrac{1}{x} + 4\right)^3$
increasing locally convex

(g) $f(x) = \displaystyle\int_x^1 e^{t^2}\,dt \quad f_1 = -e^{x^2}$
decreasing locally convex

A2.2 Find all first-order partial derivatives.

(a) $f(x_1, x_2) = 2x_1 - x_1^2 - x_2^2$
$f_1 = 2 - 2x_1 = 2(1 - x_1) \quad f_2 = -2x_2$

(b) $f(x_1, x_2) = x_1^2 + 2x_2^2 - 4x_2$
$f_1 = 2x_1 \qquad f_2 = 4x_2 - 4$

(c) $f(x_1, x_2) = x_1^3 - x_2^2 - 2x_2$

$$f_1 = 3x_1 \qquad f_2 = -2(x_2 + 1)$$

(d) $f(x_1, x_2) = 4x_1 + 2x_2 - x_1^2 + x_1x_2 - x_2^2$

$$f_1 = 4 - 2x_1 + x_2 \qquad f_2 = 2 - 2x_2 + x_1$$

(e) $f(x_1, x_2) = x_1^3 - 6x_1x_2 + x_2^3$

$$f_1 = 3x_1^2 - 6x_2 \qquad f_2 = 3x_2^2 - 6x_1$$

(f) $f(x_1, x_2) = 3x_1^2 - x_1x_2 + x_2$

$$f_1 = 6x_1 - x_2 \qquad f_2 = 1 - x_1$$

(g) $g(x_1, x_2, x_3) = ln\left(x_1^2 - x_2x_3 - x_3^2\right)$

$$g_1 = \frac{2x_1}{x_1^2 - x_2x_3 - x_3^2} \qquad g_2 = \frac{-x_3}{x_1^2 - x_2x_3 - x_3^2}$$
$$g_3 = \frac{-x_2 - 2x_3}{x_1^2 - x_2x_3 - x_3^2}$$

A2.4 Show that $y = x_1^2x_2 + x_2^2x_3 + x_3^2x_1$ satisfies the equation

$$\frac{\partial y}{\partial x_1} + \frac{\partial y}{\partial x_2} + \frac{\partial y}{\partial x_3} = (x_1 + x_2 + x_3)^2 .$$

The first-order partial derivatives are $\partial y/\partial x_1 = 2x_1x_2 + x_3^2$, $\partial y/\partial x_2 = x_1^2 + 2x_2x_3$, and $\partial y/\partial x_3 = x_2^2 + 2x_3x_1$. Summing them up gives

$$\frac{\partial y}{\partial x_1} + \frac{\partial y}{\partial x_2} + \frac{\partial y}{\partial x_3} = x_1^2 + x_2^2 + x_3^2 + 2x_1x_2 + 2x_1x_3 + 2x_2x_3 = (x_1 + x_2 + x_3)^2 .$$

A2.5 Find the Hessian matrix and construct the quadratic form, $\mathbf{z}^T\mathbf{H}(\mathbf{x})\mathbf{z}$, when

(a) $y = 2x_1 - x_1^2 - x_2^2$

$$\mathbf{H} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$$
$$\mathbf{z}^T\mathbf{H}(\mathbf{x})\mathbf{z} = -2z_1^2 + 2 * 0z_1z_2 - 2z_2^2$$

(b) $y = x_1^2 + 2x_2^2 - 4x_2$

$$\mathbf{H} = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$$
$$\mathbf{z}^T\mathbf{H}(\mathbf{x})\mathbf{z} = 2z_1^2 + 2 * 0z_1z_2 + 4z_2^2$$

(c) $y = x_1^3 - x_2^2 + 2x_2$

$$\mathbf{H} = \begin{bmatrix} 6x_1 & 0 \\ 0 & -2 \end{bmatrix}$$
$$\mathbf{z}^T\mathbf{H}(\mathbf{x})\mathbf{z} = 6x_1z_1^2 - 2z_2^2$$

(d) $y = 4x_1 + 2x_2 - x_1^2 + x_1x_2 - x_2^2$

$$\mathbf{H} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$$
$$\mathbf{z}^T\mathbf{H}(\mathbf{x})\mathbf{z} = -2z_1^2 + 2z_1z_2 - 2z_2^2$$

(e) $y = x_1^3 - 6x_1x_2 - x_2^3$

$$\mathbf{H} = \begin{bmatrix} 6x_1 & -6 \\ -6 & 6x_2 \end{bmatrix}$$
$$\mathbf{z}^T\mathbf{H}(\mathbf{x})\mathbf{z} = 6x_1z_1^2 - 12z_1z_2 + 6x_2z_2^2$$

A2.8 Suppose $f(x_1, x_2) = \sqrt{x_1^2 + x_2^2}$.
(a) Show that $f(x_1, x_2)$ is homogeneous of degree 1.
$f(tx_1, tx_2) = \sqrt{(tx_1)^2 + (tx_2)^2} = \sqrt{t^2(x_1^2 + x_2^2)} = t\sqrt{x_1^2 + x_2^2}$.
(b) According to Euler's theorem, we should have $f(x_1, x_2) = (\partial f/\partial x_1)\, x_1 + (\partial f/\partial x_2)\, x_2$. Verify this.

$$1 \cdot f(x_1, x_2) = \frac{x_1}{\sqrt{x_1^2 + x_2^2}} x_1 + \frac{x_2}{\sqrt{x_1^2 + x_2^2}} x_2 = \frac{x_1^2 + x_2^2}{\sqrt{x_1^2 + x_2^2}} = \sqrt{x_1^2 + x_2^2}$$

A2.9 Suppose $f(x_1, x_2) = (x_1x_2)^2$ and $g(x_1, x_2) = (x_1^2x_2)^3$.

(a) $f(x_1, x_2)$ is homogeneous. What is its degree?
$f(tx_1, tx_2) = t^4(x_1x_2)^2 \quad k = 4$

(b) $g(x_1, x_2)$ is homogeneous. What is its degree?
$g(tx_1, tx_2) = t^9(x_1^2x_2)^3 \quad k = 9$

(c) $h(x_1, x_2) = f(x_1, x_2)g(x_1, x_2)$ is homogeneous. What is its degree?
$h(x_1, x_2) = (x_1^3x_2^2)^5 \quad h(tx_1, tx_2) = t^{25}(x_1^3x_2^2)^5 \quad k = 25$

(d) $k(x_1, x_2) = g(f(x_1, x_2), f(x_1, x_2))$ is homogeneous. What is its degree?
$k(tx_1, tx_2) = t^{36}(x_1x_2)^{18} \quad k = 36$

(e) Prove that whenever $f(x_1, x_2)$ is homogeneous of degree m and $g(x_1, x_2)$ is homogeneous of degree n, then $k(x_1, x_2) = g(f(x_1, x_2), f(x_1, x_2))$ is homogeneous of degree mn.
$k(tx_1, tx_2) = [t^m (f(x_1, x_2), f(x_1, x_2))]^n \quad k = mn$

A2.18 Let $f(\mathbf{x})$ be a real-valued function defined on $\mathbb{R}^n_+$, and consider the matrix

$$\mathbf{H}^* = \begin{pmatrix} 0 & f_1 & \cdots & f_n \\ f_1 & f_{11} & \cdots & f_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ f_n & f_{n1} & \cdots & f_{nn} \end{pmatrix}.$$

This is a different sort of bordered Hessian than we considered in the text. Here, the matrix of second-order partials is bordered by the first-order partials and a zero to complete the square matrix. The principal minors of this matrix are the determinants

$$D_2 = \begin{vmatrix} 0 & f_1 \\ f_1 & f_{11} \end{vmatrix}, D_3 = \begin{vmatrix} 0 & f_1 & f_2 \\ f_1 & f_{11} & f_{12} \\ f_2 & f_{21} & f_{22} \end{vmatrix}, \ldots, D_n = |\mathbf{H}^*|.$$

Arrow & Enthoven (1961) use the sign pattern of these principal minors to establish the following useful results:

(i) If $f(\mathbf{x})$ is *quasiconcave*, these principal minors alternate in sign as follows: $D_2 \le 0$, $D_3 \ge 0, \ldots$.

(ii) If for all $\mathbf{x} \ge \mathbf{0}$, these principal minors (which depend on $\mathbf{x}$) alternate in sign beginning with *strictly* negative: $D_2 < 0$, $D_3 > 0, \ldots$, then $f(\mathbf{x})$ is quasiconcave on the nonnegative orthant. Further, it can be shown that if, for all $\mathbf{x} \gg \mathbf{0}$, we have this same alternating sign pattern on those principal minors, then $f(\mathbf{x})$ is *strictly* quasiconcave on the (strictly) positive orthant.

(a) The function $f(x_1, x_2) = x_1x_2 + x_1$ is quasiconcave on $\mathbb{R}^2_+$. Verify that its principal minors alternate in sign as in (ii).
Answer The bordered Hessian is

$$\mathbf{H}^* = \begin{pmatrix} 0 & x_2 + 1 & x_1 \\ x_2 + 1 & 0 & 1 \\ x_1 & 1 & 0 \end{pmatrix}.$$

The two principal minors are $D_2 = -(x_2+1)^2 < 0$ and $D_3 = 2x_1x_2+2x_1 \geq 0$. Which shows that the function will be quasiconcave and will be strictly quasiconcave for all $x_1, x_2 > 0$.

(b) Let $f(x_1, x_2) = a\ln(x_1+x_2)+b$, where $a > 0$. Is this function strictly quasiconcave for $\mathbf{x} \gg \mathbf{0}$? It is quasiconcave? How about for $\mathbf{x} \geq \mathbf{0}$? Justify.
Answer The bordered Hessian is

$$\mathbf{H}^* = \begin{pmatrix} 0 & \frac{a}{x_1+x_2} & \frac{a}{x_1+x_2} \\ \frac{a}{x_1+x_2} & \frac{-a}{(x_1+x_2)^2} & \frac{-a}{(x_1+x_2)^2} \\ \frac{a}{x_1+x_2} & \frac{-a}{(x_1+x_2)^2} & \frac{-a}{(x_1+x_2)^2} \end{pmatrix}.$$

The two principal minors are $D_2 = -(\frac{a}{x_1+x_2})^2 < 0$ for $x_1, x_2 > 0$ and $D_3 = 0$. Which shows that the function can not be strictly quasiconcave. However, it can be quasiconcave following (i). For $x_1 = x_2 = 0$ the function is not defined. Therefore, curvature can not be checked in this point.

A2.19 Let $f(x_1, x_2) = (x_1x_2)^2$. Is $f(\mathbf{x})$ concave on $\mathbb{R}^2_+$? Is it quasiconcave on $\mathbb{R}^2_+$?
Answer The bordered Hessian is

$$\mathbf{H}^* = \begin{pmatrix} 0 & 2x_1x_2^2 & 2x_1^2x_2 \\ 2x_1x_2^2 & 2x_2^2 & 4x_1x_2 \\ 2x_1^2x_2 & 4x_1x_2 & 2x_1^2 \end{pmatrix}.$$

The two principal minors are $D_2 = -(2x_1x_2)^2 < 0$ and $D_3 = 16x_1^4x_2^4 \geq 0$. Which shows that the function will be strictly quasiconcave. Strict quasiconcavity implies quasioncavity.

A2.25 Solve the following problems. State the optimised value of the function at the solution.
(a) $\min_{x_1,x_2} = x_1^2 + x_2^2$ s.t. $x_1x_2 = 1$
$x_1 = 1$ and $x_2 = 1$ or $x_1 = -1$ and $x_2 = -1$, optimised value$= 2$
(b) $\min_{x_1,x_2} = x_1x_2$ s.t. $x_1^2 + x_2^2 = 1$
$x_1 = \sqrt{1/2}$ and $x_2 = -\sqrt{1/2}$ or $x_1 = -\sqrt{1/2}$ and $x_2 = \sqrt{1/2}$, optimised value$= -1/2$
(c) $\max_{x_1,x_2} = x_1x_2^2$ s.t. $x_1^2/a^2 + x_2^2/b^2 = 1$
$x_1 = \sqrt{a^2/3}$ and $x_2 = \sqrt{2b^2/3}$ or $x_2 = -\sqrt{2b^2/3}$, optimised value$= \frac{2ab^2}{3^3/2}$
(d) $\max_{x_1,x_2} = x_1 + x_2$ s.t. $x_1^4 + x_2^4 = 1$
$x_1 = \sqrt[4]{1/2}$ and $x_2 = \sqrt[4]{1/2}$, optimised value$= \sqrt[4]{2^3} = 2^{3/4}$
(e) $\max_{x_1,x_2,x_3} = x_1x_2^2x_3^3$ s.t. $x_1 + x_2 + x_3 = 1$
$x_1 = 1/6$ and $x_2 = 1/3 = 2/6$ and $x_3 = 1/2 = 3/6$, optimised value$= 1/432 = 108/6^6$

A2.26 Graph $f(x) = 6 - x^2 - 4x$. Find the point where the function achieves its *unconstrained* (global) maximum and calculate the value of the function at that point. Compare this to the value it achieves when maximized subject to the nonnegativity constraint $x \geq 0$.

Answer This function has a global optimum at $x = -2$. It is a maximum as the second-order partial derivative is less than zero. Obviously, the global maximum is not a solution in the presence of a nonnegativity constraint. The constrained maximization problem is

$$L(x, z, \lambda) = 6 - x^2 - 4x + \lambda(x - z)$$

The first order conditions and derived equations are:

$$\begin{aligned} \frac{\partial L}{\partial x} &= -2x - 4 + \lambda = 0 & \frac{\partial L}{\partial z} &= -\lambda \leq 0 & \frac{\partial L}{\partial \lambda} &= x - z = 0 \\ \lambda &= x - z & z &= x & \lambda x &= 0 \end{aligned}$$

If $\lambda = 0$, then $x = -2$ would solve the problem. However, it does not satisfy the nonnegativity constraint. If $\lambda \neq 0$, then $x = 0$. As the function is continuously decreasing for all values $x \geq 0$, it is the only maximizer in this range.

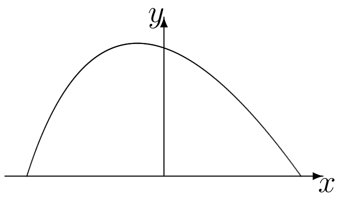


Figure 3: Graph to Exercise A2.26

2 Consumer Theory

2.1 Preferences and Utility

1.6 Cite a credible example were the preferences of an 'ordinary consumer' would be unlikely to satisfy the axiom of convexity.
Answer: Indifference curves representing satiated preferences don't satisfy the axiom of convexity. That is, reducing consumption would result in a higher utility level. Negative utility from consumption of 'bads' (too much alcohol, drugs etc.) would rather result in concave preferences.

1.8 Sketch a map of indifference sets that are parallel, negatively sloped straight lines, with preference increasing northeasterly. We know that preferences such as these satisfy Axioms $1, 2, 3$, and 4. Prove the they also satisfy Axiom $5'$. Prove that they do not satisfy Axiom 5.
Answer: Definition of convexity (Axiom $5'$): If $\mathbf{x}^1 \succsim \mathbf{x}^0$, then $t\mathbf{x}^1 + (1-t)\mathbf{x}^0 \succsim \mathbf{x}^0$ for all $t \in [0, 1]$. Strict convexity (Axiom 5) requires that, if $\mathbf{x}^1 \neq \mathbf{x}^0$ and $\mathbf{x}^1 \succsim \mathbf{x}^0$, then $t\mathbf{x}^1 + (1-t)\mathbf{x}^0 \succ \mathbf{x}^0$ for all $t \in [0, 1]$.
The map of indifference sets in the figure below represent perfect substitues. We know that those preferences are convex but not stricly convex. Intuitively, all combinations of two randomly chosen bundles from one indifference curve will necessarily lie on the same indifference curve. Additionally, the marginal rate of substitution does not change by moving from $\mathbf{x}^0$ to $\mathbf{x}^1$. To prove the statement more formally, define $\mathbf{x}^t$ as convex combination of bundles $\mathbf{x}^0$ to $\mathbf{x}^1$: $\mathbf{x}^t = t\mathbf{x}^0 + (1-t)\mathbf{x}^1$. Re-writing in terms of single commodities gives us:
$\mathbf{x}^t = (tx_1^0, tx_2^0) + ((1-t)x_1^1, (1-t)x_2^1)$. A little rearrangement and equalising the two definitions results in the equality
$t\mathbf{x}^0 + (1-t)\mathbf{x}^1 = (tx_1^0 + (1-t)x_1^1), tx_2^0 + (1-t)x_2^1)$. That is, the consumer is indifferent with respect to the convex combination and the original bundles, a clear violation of strict convexity.

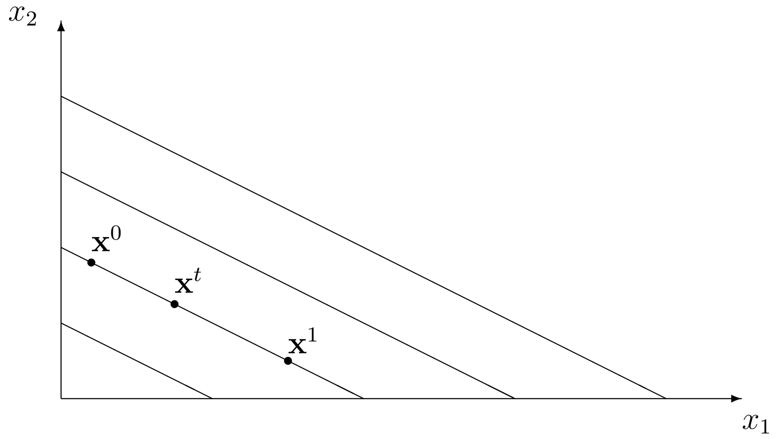


Figure 4: Indifference sets to Exercise 1.8

1.9 Sketch a map of indifference sets that are parallel right angles that "kink" on the line $x_1 = x_2$. If preference increases northeasterly, these preferences will satisfy Axioms 1, 2, 3, and 4'. Prove that they also satisfy Axiom 5'. Do they also satisfy Axiom 4? Do they satisfy Axiom 5?

Answer: Convexity (Axiom 5′) requires that, if $\mathbf{x}^1 \succsim \mathbf{x}^0$, then $t\mathbf{x}^1 + (1-t)\mathbf{x}^0 \succsim \mathbf{x}^0$ for all $t \in [0, 1]$.

Take any two vectors $\mathbf{x}^0$, $\mathbf{x}^1$ such that $\mathbf{x}^0 \sim \mathbf{x}^1$. Given the nature of these preferences, it must be true that $\min[x_1^0, x_2^0] = \min[x_1^1, x_2^1]$. For any $t \in [0, 1]$ consider the point $tx_1 + (1-t)x_2$. If we can show that $\min[tx_1^0 + (1-t)x_2^0, tx_1^1 + (1-t)x_2^1] \geq \min[x_1^0, x_2^0 = \min[x_1^1, x_2^1]$, then we shown that these preferences are convex. $\min[tx_1^0 + (1-t)x_2^0, tx_1^1 + (1-t)x_2^1] \geq \min[tx_1^0, tx_1^1] + \min[(1-t)x_2^0, +(1-t)x_2^1] = \min[x_2^0, x_2^1] + t[\min(x_1^0, x_1^1) - \min(x_2^0, x_2^1)] = \min[x_2^0, x_2^1]$

Definition of strict monotonicity (Axiom 4): For all $\mathbf{x}^0$, $\mathbf{x}^1 \in \mathbb{R}_+^n$, if $\mathbf{x}^0 \geq \mathbf{x}^1$, then $\mathbf{x}^0 \succsim \mathbf{x}^1$, while if $\mathbf{x}^0 \gg \mathbf{x}^1$, then $\mathbf{x}^0 \succ \mathbf{x}^1$.

The map of indifference sets in the figure below represents perfect complements. Take two points $\mathbf{x}^0, \mathbf{x}^1$ along one indifference curve. If $\mathbf{x}^0 \gg \mathbf{x}^1$, "preferences increase northeasterly", then $\mathbf{x}^0 \succ \mathbf{x}^1$. For any two vectors on the same indifference curve, that is $\mathbf{x}^0 \geq \mathbf{x}^1$, it follows $\mathbf{x}^0 \succsim \mathbf{x}^1$. Therefore, the definition of strict monotonicity is satisfied for these indifference sets.

Strict convexity (Axiom 5) requires that, if $\mathbf{x}^1 \neq \mathbf{x}^0$ and $\mathbf{x}^1 \succsim \mathbf{x}^0$, then $t\mathbf{x}^1 + (1-t)\mathbf{x}^0 \succ \mathbf{x}^0$ for all $t \in [0, 1]$.

Take any two points along the horizontal or vertical part of an indifference curve such as (x_1^0, x_2^0) and (x_1^0, x_2^1), where $x_2^0 > x_2^1$. Any convex combination $\mathbf{x}^t = x_1^0, tx_2^0 + (1-t)x_2^1$ lies on the same indifference curve as $\mathbf{x}^1$ and $\mathbf{x}^0$. Therefore, it is not possible that $\mathbf{x}^t \succ t\mathbf{x}^0 + (1-t)\mathbf{x}^1$. That is, the consumer is indifferent with respect to the convex combination and the original bundles, a clear violation of strict convexity.

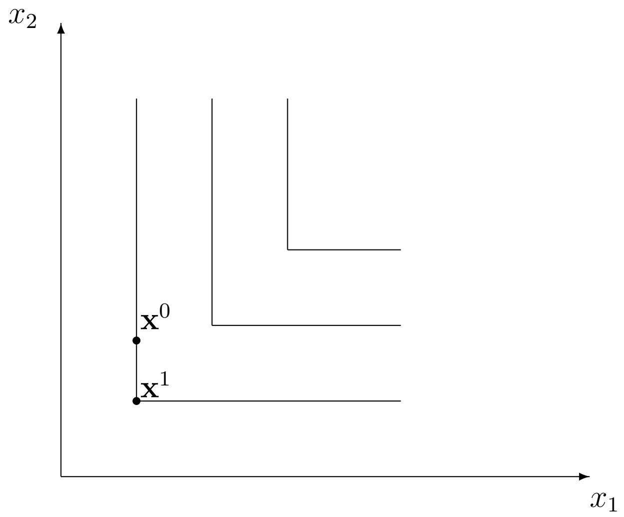


Figure 5: Indifference sets to Exercise 1.9

1.12 Suppose $u(x_1, x_2)$ and $v(x_1, x_2)$ are utility functions.

(a) Prove that if $u(x_1, x_2)$ and $v(x_1, x_2)$ are both homogeneous of degree r, then $s(x_1, x_2) \equiv u(x_1, x_2) + v(x_1, x_2)$ is homogeneous of degree r.
Answer: Whenever it holds that $t^r u(x_1, x_2) = u(tx_1, tx_2)$ and $t^r v(x_1, x_2) = v(tx_1, tx_2)$ for all $r > 0$, it must also hold that $t^r s(x_1, x_2) \equiv u(tx_1, tx_2) + v(tx_1, tx_2) = t^r u(x_1, x_2) + t^r v(x_1, x_2)$.

(b) Prove that if $u(x_1, x_2)$ and $v(x_1, x_2)$ are quasiconcave, then $m(x_1, x_2) \equiv u(x_1, x_2) + v(x_1, x_2)$ is also quasiconcave.
Answer: Forming a convex combination of the two functions u and v and comparing with $m(\mathbf{x}^t)$ satisfies the definition of quasiconcavity:

$$\begin{aligned}
\text{When } u(\mathbf{x}^t) &\geq \min\left\{tu(\mathbf{x}^1) + (1-t)u(\mathbf{x}^2)\right\} \text{ and} \\
v(\mathbf{x}^t) &\geq \min\left\{tv(\mathbf{x}^1) + (1-t)v(\mathbf{x}^2)\right\} \text{ so} \\
m(\mathbf{x}^t) &\geq \min\left\{u(\mathbf{x}^t) + v(\mathbf{x}^t)\right\} \\
&= \left[t(u(\mathbf{x}^1) + v(\mathbf{x}^1)) + (1-t)(u(\mathbf{x}^2) + v(\mathbf{x}^2))\right].
\end{aligned}$$

2.2 The Consumer's Problem

1.20 Suppose preferences are represented by the Cobb-Douglas utility function, $u(x_1, x_2) = Ax_1^{\alpha}x_2^{1-\alpha}$, $0 < \alpha < 1$, and $A > 0$. Assuming an interior solution, solve for the Marshallian demand functions.
Answer: Use either the Lagrangian or the equality of Marginal Rate of Substitution and price ratio. The Lagrangian is

$L = Ax_1^{\alpha}x_2^{1-\alpha} + \lambda(y - p_1x_1 - p_2x_2)$. The first-order conditions (FOC) are

$$\frac{\partial L}{\partial x_1} = \alpha A x_1^{\alpha-1}x_2^{1-\alpha} - \lambda p_1 = 0$$
$$\frac{\partial L}{\partial x_2} = (1-\alpha)Ax_1^{\alpha}x_2^{-\alpha} - \lambda p_2 = 0$$
$$\frac{\partial L}{\partial \lambda} = y - p_1x_1 + p_2x_2 = 0$$

By dividing first and second FOC and some rearrangement, we get either $x_1 = \frac{\alpha x_2 p_2}{(1-\alpha)p_1}$ or $x_2 = \frac{(1-\alpha)p_1x_1}{\alpha p_2}$. Substituting one of these expressions into the budget constraint, results in the Marshallian demand functions: $x_1 = \frac{\alpha y}{p_1}$ and $x_2 = \frac{(1-\alpha)y}{p_2}$.

1.21 We've noted that $u(\mathbf{x})$ is invariant to positive monotonic transforms. One common transformation is the logarithmic transform, $\ln(u(\mathbf{x}))$. Take the logarithmic transform of the utility function in 1.20; then, using that as the utility function, derive the Marshallian demand functions and verify that they are identical to those derived in the preceding exercise (1.20).
Answer: Either the Lagrangian is used or the equality of Marginal Rate of Substitution with the price ratio. The Lagrangian is
$L = \ln(A) + \alpha\ln(x_1) + (1-\alpha)\ln(x_2) + \lambda(y - p_1x_1 - p_2x_2)$. The FOC are

$$\frac{\partial L}{\partial x_1} = \frac{\alpha}{x_1} - \lambda p_1 = 0$$
$$\frac{\partial L}{\partial x_2} = \frac{(1-\alpha)}{x_2} - \lambda p_2 = 0$$
$$\frac{\partial L}{\partial \lambda} = y - p_1x_1 + p_2x_2 = 0$$

The Marshallian demand functions are: $x_1 = \frac{\alpha y}{p_1}$ and $x_2 = \frac{(1-\alpha)y}{p_2}$. They are exactly identical to the demand functions derived in the preceding exercise.

1.24 Let $u(\mathbf{x})$ represent some consumer's monotonic preferences over $\mathbf{x} \in \mathbb{R}^n_+$. For each of the functions $F(\mathbf{x})$ that follow, state whether or not f also represents the preferences of this consumer. In each case, be sure to justify your answer with either an argument or a counterexample.
Answer:

(a) $f(\mathbf{x}) = u(\mathbf{x}) + (u(\mathbf{x}))^3$ Yes, all arguments of the function u are transformed equally by the third power. Checking the first- and second-order partial derivatives reveals that, although the second-order partial $\frac{\partial^2 f}{\partial x_i^2} = \frac{\partial^2 u}{\partial x_i^2} + 6(u(\mathbf{x}))(\frac{\partial u}{\partial x_i})^2$ is not zero, the sign of the derivatives is always invariant and positive. Thus, f represents a monotonic transformation of u.

(b) $f(\mathbf{x}) = u(\mathbf{x}) - (u(\mathbf{x}))^2$ No, function f is decreasing with increasing consumption for any $u(\mathbf{x}) < (u(\mathbf{x}))^2$. Therefore, it can not represent the preferences of the consumer. It could do so if the minus sign is replaced by a plus sign.

(c) $f(\mathbf{x}) = u(\mathbf{x}) + \sum_{i=1}^{n} x_i$ Yes, the transformation is a linear one, as the first partial is a positive constant, here one, and the second partial of the transforming function is zero. Checking the partial derivatives proves this statement: $\frac{\partial f}{\partial x_i} = \frac{\partial u}{\partial x_i} + 1$ and $\frac{\partial^2 f}{\partial x_i^2} = \frac{\partial^2 u}{\partial x_i^2}$.

1.28 An infinitely lived agent owns 1 unit of a commodity that she consumes over her lifetime. The commodity is perfect storable and she will receive no more than she has now. Consumption of the commodity in period t is denoted x_t, and her lifetime utility function is given by

$$u(x_0, x_1, x_2, \ldots) = \sum_{t=0}^{\infty} \beta^t \ln(x_t), \text{ where } 0 < \beta < 1.$$

Calculate her optimal level of consumption in each period.
Answer: Establish a geometric series to calculate her lifetime utility:

$$u = \beta^0 \ln(x_0) + \beta \ln(x_1) + \beta^2 \ln(x_2) + \ldots + \beta^t \ln(x_t)$$

As β is less than one, this series approaches a finite value. To find the solution, multiply the expression by β and subtract from the original equation [(1)-(2)].

$$\begin{aligned} \beta u &= \beta^1 \ln(x_0) + \beta^2 \ln(x_1) + \beta^3 \ln(x_2) + \ldots + \beta^{t+1} \ln(x_t) \\ u - \beta u &= (1-\beta)u = \ln(x_0) - \beta^{t+1} \ln(x_t) \\ u &= \frac{\ln(x_0) - \beta^{t+1} \ln(x_t)}{1-\beta} = \ln(x_0) \end{aligned}$$

Thus, the consumer's utility maximising consumption will be constant in every period.

2.3 Indirect Utility and Expenditure

1.30 Show that the indirect utility function in Example 1.2 is a quasi-convex function of prices and income.
Answer: The indirect utility function corresponding to CES preferences is: $v(\mathbf{p}, y) = y\left(p_1^r + p_2^r\right)^{-1/r}$, where $r \equiv \rho/(\rho - 1)$.
There are several ways. First, using the inequality relationship, let $\mathbf{p}^t = t\mathbf{p}^0 + (1-t)\mathbf{p}^1$. We need to show that the indirect utility function fulfills the inequality

$$y\left(p_1^{tr} + p_2^{tr}\right)^{-1/r} \leq \max[y\left(p_1^{0r} + p_2^{0r}\right)^{-1/r}, y\left(p_1^{1r} + p_2^{1r}\right)^{-1/r}]$$

which gives:

$$y\left(t^r(p_1^{0r}+p_2^{0r})+(1-t)^r(p_1^{1r}+p_2^{1r})\right)^{-1/r}\le \max[y\left(p_1^{0r}+p_2^{0r}\right)^{-1/r},y\left(p_1^{1r}+p_2^{1r}\right)^{-1/r}].$$

Second, the bordered Hessian can be derived and their determinants checked. The determinants will be all negative.

$$H=\begin{pmatrix} 0 & \mathbf{p}^{-1/r} & -p_1^{r-1}\mathbf{p}^{-1/r-1}y & -p_2^{r-1}\mathbf{p}^{-1/r-1}y \\ \mathbf{p}^{-1/r} & 0 & \mathbf{p}^{-1/r} & \mathbf{p}^{-1/r} \\ -p_1^{r-1}\mathbf{p}^{-1/r-1}y & \mathbf{p}^{-1/r} & y\mathbf{p}^{-1/r-1}p_1^{r-2}\left((1-r)-rp_1^r\mathbf{p}^{-1}\right) & (1+r)p_1^{r-1}p_2^{r-1}\mathbf{p}^{-1/r-2}y \\ -p_2^{r-1}\mathbf{p}^{-1/r-1}y & \mathbf{p}^{-1/r} & (1+r)p_1^{r-1}p_2^{r-1}\mathbf{p}^{-1/r-2}y & y\mathbf{p}^{-1/r-1}p_2^{r-2}\left((1-r)-rp_2^r\mathbf{p}^{-1}\right) \end{pmatrix}$$

, where $\mathbf{p}\equiv(p_1^r+p_2^r)$.

1.37 Verify that the expenditure function obtained from the CES direct utility function in Example 1.3 satisfies all the properties given in Theorem 1.7.
Answer: The expenditure function for two commodities is $e(\mathbf{p},u)=u\left(p_1^r+p_2^r\right)^{1/r}$ where $r\equiv\rho/(\rho-1)$.

1. Zero when u takes on the lowest level of utility in U.
   The lowest value in U is $u((0))$ because the utility function is strictly increasing. Consequently, $0(p_1^r+p_2^r)^{1/r}=0$.

2. Continuous on its domain $\mathbb{R}^n_{++}\times U$.
   This property follows from the Theorem of Maximum. As the CES direct utility function satisfies the axiom of continuity, the derived expenditure function will be continuous too.

3. For all $\mathbf{p}>>\mathbf{0}$, strictly increasing and unbounded above in u.
   Take the first partial derivative of the expenditure function with respect to utility: $\partial e/\partial u=(p_1^r+p_2^r)^{1/r}$. For all strictly positive prices, this expression will be positive. Alternatively, by the Envelope theorem it is shown that the partial derivative of the minimum-value function e with respect to u is equal to the partial derivative of the Lagrangian with respect to u, evaluated at $(\mathbf{x}^*,\lambda^*)$, what equals λ. Unboundness above follows from the functional form of u.

4. Increasing in $\mathbf{p}$.
   Again, take all first partial derivatives with respect to prices: $\partial e/\partial p_i=up_i^{r-1}\left(p_1^r+p_2^r\right)^{(1/r)-1}$, what is, obviously, positive.

5. Homogeneous of degree 1 in $\mathbf{p}$.
   $e(t\mathbf{p},u)=u\left((tp_1)^r+(tp_2)^r\right)^{1/r}=t^1u\left(p_1^r+p_2^r\right)^{1/r}$

6. Concave in $\mathbf{p}$.
   The definition of concavity in prices requires

$$t\left[u\left({p^0}_1^r+{p^0}_2^r\right)^{1/r}\right]+(1-t)\left[u\left({p^1}_1^r+{p^1}_2^r\right)^{1/r}\right]\leq e(\mathbf{p}^t,u)$$

   for $\mathbf{p}^t=t\mathbf{p}^0+(1-t)\mathbf{p}^1$. Plugging in the definition of the price vector into $e(\mathbf{p}^t,u)$ yields the relationship

$$t\left[u\left({p^0}_1^r+{p^0}_2^r\right)^{1/r}\right]+(1-t)\left[u\left({p^1}_1^r+{p^1}_2^r\right)^{1/r}\right]\leq \\ u\left(t({p^0}_1^r+{p^0}_2^r)+(1-t)({p^1}_1^r+{p^1}_2^r)\right)^{1/r}.$$

   Alternatively, we can check the negative semidefiniteness of the associated Hessian matrix of all second-order partial derivatives of the expenditure function. A third possibility is to check (product rule!)

$$\frac{\partial^2 e}{\partial p_i^2}=u\left((r-1)\frac{p_i^r(p_1^r+p_2^r)^{1/r}}{p_1^2(p_1^r+p_2^r)}-r\frac{p_i^{2r}(p_1^r+p_2^r)^{1/r}}{p_i^2(p_1^r+p_2^r)^2}\right)<0 \text{ by } r<0.$$

   Homogeneity of degree one, together with Euler's theorem, implies that $\partial^2 e/\partial p_i^2 p_i=0$. Hence the diagonal elements of the Hessian matrix must be zero and the matrix will be negative semidefinite.

7. Shephard's lemma
   $\partial e/\partial u=(p_1^r+p_2^r)^{1/r}$ what is exactly the definition of a CES-type Hicksian demand function.

1.38 Complete the proof of Theorem 1.9 by showing that
$\mathbf{x}^h(\mathbf{p},u)=x\left(\mathbf{p},e(\mathbf{p},u)\right)$.
Answer: We know that at the solution of the utility maximisation or expenditure minimisation problem $e(\mathbf{p},u)=y$ and $u=v(\mathbf{p},y)$. Substitute the indirect utility function v into the Hicksian demand function gives $\mathbf{x}^h(\mathbf{p},v(\mathbf{p},y))$. As the new function is a function of prices and income only, it is identical to the Marshallian demand function. Furthermore, by replacing income by the expenditure function we get the expression $x\left(\mathbf{p},e(\mathbf{p},u)\right)$.

2.4 Properties of Consumer Demand

1.40 Prove that Hicksian demands are homogeneous of degree zero in prices.
Answer: We know that the expenditure function must be homogeneous of degree one in prices. Because any Hicksian demand function equals, due to Shephard's lemma, the first partial derivative of the expenditure function and, additionally, we know that the derivative's degree of homogeneity is k-1. The Hicksian demand functions must be homogeneous of degree $1-1=0$ in prices.

1.43 In a two-good case, show that if one good is inferior, the other good must be normal.
Answer: The Engel-aggregation in a two-good case is the product of the income elasticity and the repsective expenditure share $s_1\eta_1 + s_2\eta_2 = 1$. An inferior good is characterised by a negative income elasticity, thus, one of the two summands will be less than zero. Therefore, to secure this aggregation, the other summand must be positive (even larger one) and the other commodity must be a normal good (even a luxury item).

1.55 What restrictions must the $\alpha_i, f(y), w(p_1,p_2)$, and $z(p_1,p_2)$ satisfy if each of the following is to be a legitimate indirect utility function?
Answer:
(a) $v(p_1,p_2,p_3,y) = f(y)p_1^{\alpha_1}p_2^{\alpha_2}p_3^{\alpha_3}$ The function $f(y)$ must be continuous, strictly increasing and homogeneous of degree $0 - \sum\alpha_i$. Each of the exponents α_i has to be less than zero to satisfy v decreasing in prices. Furthermore, negative partial derivatives of v with respect to each price are required to get positive Marshallian demand functions by using Roy's identity.
(b) $v(p_1,p_2,y) = w(p_1,p_2)+z(p_1,p_2)/y$ The functions w and z must be continuous and decreasing in prices. Function z has to be homogeneous of degree one and w homogeneous of degree zero: $v(tp_1,tp_2,ty) = t^0w(p_1,p_2)+(t^1z(p_1,p_2))/(ty) = t^0\left(w(p_1,p_2)+z(p_1,p_2)/y\right)$. To satisfy v increasing in income, z must be < 0.

1.60 Show that the Slutsky relation can be expressed in elasticity form as $\epsilon_{ij} = \epsilon_{ij}^h - s_j\eta_i$, where ϵ_{ij}^h is the elasticity of the Hicksian demand for x_i with respect to price p_j, and all other terms are as defined in Definition 1.6. *Answer*: The Slutsky relation is given by

$$\frac{\partial x_i}{\partial p_j} = \frac{\partial x_i^h}{\partial p_j} - x_j\frac{\partial x_i}{\partial y}.$$

Multiplying the total expression with y/y and p_j gives

$$\frac{\partial x_i}{\partial p_j}p_j = \frac{\partial x_i^h}{\partial p_j}p_j - \frac{p_jx_j}{y}\frac{\partial x_i}{\partial y}y.$$

By assuming that $x_i^h = x_i$ before the price change occurs, we can divide all three terms by x_i. The result of this operation is

$$\frac{\partial x_i}{\partial p_j}\frac{p_j}{x_i} = \frac{\partial x_i^h}{\partial p_j}\frac{p_j}{x_i} - s_j\frac{\partial x_i}{\partial y}\frac{y}{x_i} = \epsilon_{ij} = \epsilon_{ij}^h - s_j\eta_i$$

Additional exercise Relationship between utility maximisation and expenditure minimisation
Let's explore the relationship with an example of a concrete utility function. A consumer's utility function is $u = x_1^{1/2}x_2^{1/2}$. For the derived functions see 1

<table>
<tr><td>Start from the utility function
and derive the Marshallian demand for x_1
$x_1 = y/2p_1$</td><td>Minimise expenditures s.t. u
to find the Hicksian demand function
$x_1^h = u\,(p_2/p_1)^{1/2}$</td></tr>
<tr><td colspan="2">Plug in the respective demand functions to get the</td></tr>
<tr><td>indirect utility function
$v = y/(4p_1p_2)^{1/2}$</td><td>expenditure function
$e = u(4p_1p_2)^{1/2}$</td></tr>
<tr><td>Substitute the expenditure function
into the Marshallian demand function
to derive the Hicksian demand function
$x_1 = (u(4p_1p_2)^{1/2})/2p_1 = u(p_2/p_1)^{1/2}$</td><td>Substituting the indirect utility function
into the Hicksian demand function
to derive the Marshallian demand function
$x_1^h = (p_2/p_1)^{1/2}y/(4p_1p_2)^{1/2} = y/2p_1$</td></tr>
<tr><td>Invert v and replace y by u
to get the expenditure function
$v^{-1} = u(4p_1p_2)^{1/2}$</td><td>Invert e and replace u by v
to get the indirect utility function
$e^{-1} = y(4p_1p_2)^{-1/2}$</td></tr>
<tr><td>Check Roy's identity
$-\frac{\partial v/\partial p_1}{\partial v/\partial y} = \frac{2y(p_1p_2)^{1/2}}{4(p_1^3p_2)^{1/2}} = y/2p_1$</td><td>Check Shephard's lemma
$\frac{\partial e}{\partial p_1} = \frac{u4p_2}{2(4p_1p_2)^{1/2}} = u(p_2/p_1)^{1/2}$</td></tr>
<tr><td colspan="2">Establish the Slutsky equation
$\frac{\partial x_1}{\partial p_2} = \frac{u}{2(p_1p_2)^{1/2}} - \frac{y}{2p_2} \cdot \frac{1}{2p_1}$
substitute $u = v(\mathbf{p}, y)$ into the substitution effect
$\frac{\partial x_1}{\partial p_2} = \frac{y}{4p_1p_2} - \frac{y}{4p_1p_2} = 0$</td></tr>
</table>

Table 1: Relationship between UMP and EMP

2.5 Equilibrium and Welfare

4.19 A consumer has preferences over the single good x and all other goods m represented by the utility function, $u(x, m) = \ln(x) + m$. Let the price of x be p, the price of m be unity, and let income be y.

(a) Derive the Marshallian demands for x and m.
Answer The equality of marginal rate of substitution and price ratio gives $1/x = p$. Thus, the Marshallian demand for x is $x = 1/p$. The uncompensated demand for m separates into two cases depending on the amount of income available:

$$m = \begin{cases} 0 & \text{when } y \leq 1 \\ y - 1 & \text{when } y > 1. \end{cases}$$

(b) Derive the indirect utility function, $v(p, y)$.
Answer Again, depending on the amount of income available there will be two indirect utility functions:

$$v(p, y) = \begin{cases} \ln\left(\frac{1}{p}\right) & \text{when } m \leq 1 \\ y - 1 - \ln p & \text{when } m > 1. \end{cases}$$

(c) Use the Slutsky equation to decompose the effect of an own-price change on the demand for x into an income and substitution effect. Interpret your result briefly.
Answer A well-known property of any demand function derived from a quasi-linear utility function is the absence of the income effect. Which can be easily seen in the application of the Slutsky equation:

$$\frac{\partial x^h}{\partial p} = \frac{\partial x}{\partial p} + x\frac{\partial x}{\partial y}$$

$$\frac{\partial x}{\partial p} = -\frac{1}{p^2} + 0 \cdot \frac{1}{p} = \frac{\partial x^h}{\partial p}.$$

Therefore, the effect of an own-price change on the demand for x equals the substitution effect.

(d) Suppose that the price of x rises from p^0 to $p^1 > p^0$. Show that the consumer surplus area between p^0 and p^1 gives an *exact* measure of the effect of the price change on consumer welfare.
Answer The consumer surplus area can be calculated by integrating over the inverse demand function of x:

$$CS = \int_{p^0}^{p^1} \frac{1}{x}\mathrm{d}x = \ln(p^1 - p^0).$$

Calculating the change in utility induced by a price change gives:

$$\Delta v = v^1(p^1, y^1) - v^0(p^1, y^0) = y - 1 - \ln p^1 - (y - 1 - \ln p^0) = \ln(p^1 - p^0).$$

As the two expressions are equal, the consumer surplus area gives an exact measure of the effect of the price change on consumer welfare in the case of quasi-linear preferences.

(e) Carefully illustrate your findings with a set of *two* diagrams: one giving the indifference curves and budget constraints on top, and the other giving the Marshallian and Hicksian demands below. Be certain that your diagrams reflect all qualitative information on preferences and demands that you've uncovered. Be sure to consider the two prices p^0 and p^1, and identify the Hicksian and Marshallian demands.
Answer See Figure 6. Please note, that Hicksian and Marshallian demands are identical here.

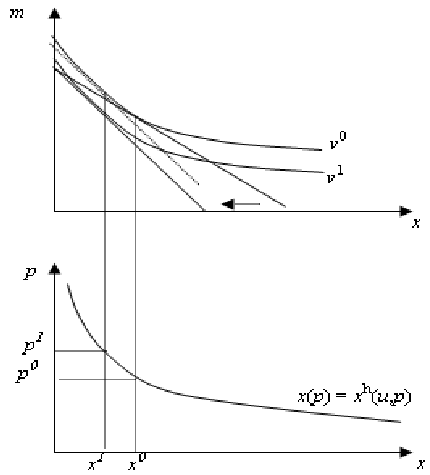


Figure 6: Graph to 4.19

3 Producer Theory

3.1 Production

3.1 The elasticity of average product is defined as $\frac{\partial AP_i(\mathbf{x})}{\partial x_i} \cdot \frac{x_i}{AP_i(\mathbf{x})}$. Show that this is equal to $\mu_i(\mathbf{x}) - 1$. Show that average product is increasing, constant, or decreasing as marginal product exceeds, is equal to, or less than average product.
Answer: Applying quotient rule to get the first partial derivative of the average product gives:

$$\frac{\partial AP_i(\mathbf{x})}{\partial x_i} = \frac{x_i \partial f(\mathbf{x})/\partial x_i - f(\mathbf{x})}{x_i^2} = \frac{MP}{x_i} - \frac{AP}{x_i} = \frac{MP - AP}{x_i}$$

Multiply this term with the right part of the definition (x_i/AP) gives $MP/AP - 1$ what is exactly $\mu_i(\mathbf{x}) - 1$.
The first part of the above definition equals the slope of the average product: $(MP - AP)/x_i$. It is straightforward to show that whenever marginal product exceeds the average product the slope has to be positive. The average product reaches a maximum when the marginal product equals average product. Finally, whenever $MP < AP$ average product is sloping downwards.

3.3 Prove that when the production function is homogeneous of degree one, it may be written as the sum $f(\mathbf{x}) = \sum MP_i(\mathbf{x})x_i$, where $MP_i(\mathbf{x})$ is the marginal product of input i.
Answer: The answer to this exercise gives a nice application of Euler's Theorem. The sum of the partial differentials of a function multiplied with the level of the respective inputs is equal to the function times the degree of homogeneity k. The sum of all marginal products multiplied with input levels gives the production function times $k = 1$.

3.7 Goldman & Uzawa (1964) have shown that the production function is weakly separable with respect to the partition $\{N_1, \ldots, N_S\}$ if and only if it can be written in the form

$$f(\mathbf{x}) = g\left(f^1(\mathbf{x}^{(1)}), \ldots, f^S(\mathbf{x}^{(S)})\right),$$

where g is some function of S variables, and, for each i, $f^i(\mathbf{x}^{(i)})$ is a function of the subvector $x^{(i)}$ of inputs from group i alone. They have also shown that the production function will be strongly separable if and only if it is of the form

$$f(\mathbf{x}) = G\left(f^1(\mathbf{x}^{(1)}) + \cdots + f^S(\mathbf{x}^{(S)})\right),$$

where G is a strictly increasing function of one variable, and the same conditions on the subfunctions and subvectors apply. Verify their results by showing that each is separable as they claim.
Answer To show that the first equation is weakly separable with respect to the partitions, we need to show that $\frac{\partial[f_i(x)/f_j(x)]}{\partial x_k} = 0 \; \forall i, j \in N_S$ and $k \notin N_S$. Calculate the marginal

products of the first equation for two arbitrary inputs i and j:

$$f_i(\mathbf{x}) = \frac{\partial g}{\partial f^S}\frac{\partial f^S}{\partial x_i} \qquad f_j(\mathbf{x}) = \frac{\partial g}{\partial f^S}\frac{\partial f^S}{\partial x_j}.$$

The marginal rate of technical substitution between these two inputs is

$$\frac{f_i(\mathbf{x})}{f_j(\mathbf{x})} = \frac{\frac{\partial f^S}{\partial x_i}}{\frac{\partial f^S}{\partial x_j}}$$

This expression is independent of any other input which is not in the same partition N^S and, therefore, the production function is weakly separable.

$$\frac{\partial(f_i/f_j)}{\partial x_k} = 0 \text{ for } k \notin N^S$$

To show that the second equation is strongly separable we have to perform the same exercise, however, assuming that the three inputs are elements of three different partitions $i \in N_S, j \in N_T$ and $k \notin N_S \cup N_T$. The marginal products of the two inputs i and j are:

$$f_i(\mathbf{x}) = G'\frac{\partial f^S\left(x^{(S)}\right)}{\partial x_i} \qquad f_j(\mathbf{x}) = G'\frac{\partial f^T\left(x^{(T)}\right)}{\partial x_j}.$$

The MRTS is:

$$\frac{f_i(\mathbf{x})}{f_j(\mathbf{x})} = \frac{\partial f^S/\partial x_i}{\partial f^T/\partial x_j}.$$

It follows for $k \notin N_S \cup N_T$

$$\frac{\partial(f_i/f_j)}{\partial x_k} = 0.$$

3.8 A Leontief production function has the form $y = \min\{\alpha x_1, \beta x_2\}$ for $\alpha > 0$ and $\beta > 0$. Carefully sketch the isoquant map for this technology and verify that the elasticity of substitution $\sigma = 0$, where defined.
Answer: Taking the total differential of the log of the factor ratio gives $\mathrm{d}\ln(\beta x_2/\alpha x_1) = \beta/x_2\mathrm{d}x_2 - \alpha/x_1\mathrm{d}x_1$. However, the MRTS is not defined in the kinks as the function is discontinuous. Along all other segments of the isoquants the MRTS is zero. Therefore, the elasticity of substitution is only defined when the input ratio remains constant. In this case, $\sigma = 0$.

3.9 Calculate σ for the Cobb-Douglas production function $y = Ax_1^\alpha x_2^\beta$, where $A > 0, \alpha > 0$ and $\beta > 0$.
Answer: The total differential of the log of the factor ratio gives
$\mathrm{d}\ln(x_2/x_1) = \beta/x_2\mathrm{d}x_2 - \alpha/x_1\mathrm{d}x_1$. The total differential of the marginal rate of technical substitution gives

$$\mathrm{d}\ln\left(\frac{A\alpha x_1^{\alpha-1}x_2^\beta}{A\beta x_1^\alpha x_2^{\beta-1}}\right) = \alpha/\beta(\mathrm{d}x_1/x_1 - \mathrm{d}x_2/x_2)$$

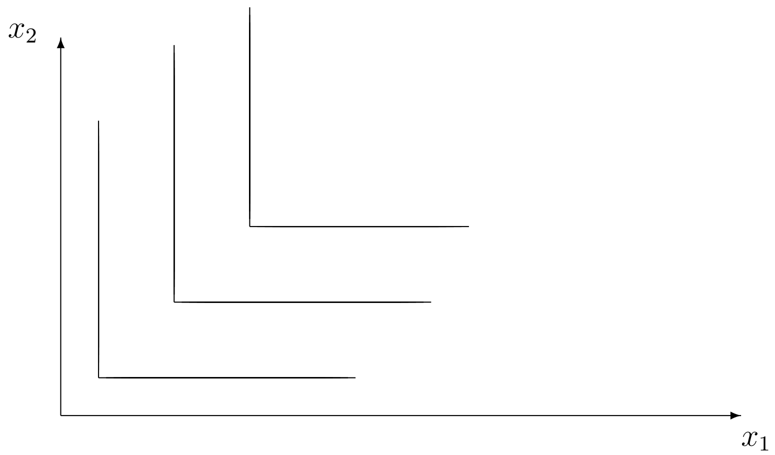


Figure 7: Isoquant map of Leontief technology

Putting both parts together results in

$$\sigma = \frac{\beta/x_2 \mathrm{d}x_2 - \alpha/x_1 \mathrm{d}x_1}{\alpha/\beta(\mathrm{d}x_1/x_1 - \mathrm{d}x_2/x_2)} = 1$$

3.14 Let $y = (\sum_{i=1}^{n} \alpha_i x_i^{\rho})^{1/\rho}$, where $\sum_i \alpha_i = 1$ and $0 \neq \rho < 1$. Verify that $\sigma_{ij} = 1/(1-\rho)$ for all $i \neq j$.
Answer Apply the definition of the elasticity of substitution.

$$\begin{aligned}
\sigma_{ij} &= \frac{\partial\left(\ln(x_j) - \ln(x_i)\right)}{\partial \ln\left(f_i(x)/f_j(x)\right)} \\
&= \frac{\frac{1}{x_j}\partial x_j - \frac{1}{x_i}\partial x_i}{\partial \ln\left(\frac{\alpha_i x_i^{\rho-1}(\sum_i \alpha_i x_i^{\rho})^{1/\rho-1}}{\alpha_j x_j^{\rho-1}(sum_i \alpha_i x_i^{\rho})^{1/\rho-1}}\right)} \\
&= \frac{-\left(\frac{1}{x_i}\partial x_i - \frac{1}{x_j}\partial x_j\right)}{\rho - 1\left(\frac{1}{x_i}\partial x_i - \frac{1}{x_j}\partial x_j\right)} \\
&= \frac{-1}{\rho-1} = \frac{1}{1-\rho}
\end{aligned}$$

3.15 For the generalised CES production function, prove the following claims made in the text.

$$y = \left(\sum_{i=1}^{n} \alpha_i x_i^{\rho}\right)^{1/\rho}, \text{ where } \sum_{i=1}^{n} \alpha_i = 1 \text{ and } 0 \neq \rho < 1$$

(a)

$$\lim_{\rho \to 0} y = \prod_{i=1}^{n} x_i^{\alpha_i}$$

Answer: Write the log of the CES production function $\ln y = 1/\rho \ln \sum \alpha_i x_i^\rho$. At $\rho = 0$, the value of the function is indeterminate. However, using L'Hòpital's rule we can write

$$\lim_{\rho \to 0} \ln y = \frac{\sum \alpha_i x_i^\rho \ln x_i}{\sum \alpha_i x_i^\rho}.$$

At $\rho = 0$ this expression turns into $\ln y = \sum \alpha_i \ln x_i / \sum \alpha_i$. Because the denominator is defined to be one, we can write the CES production at this point as $y = \prod x_i^{\alpha_i}$, what is exactly the generalised Cobb-Douglas form.

(b)

$$\lim_{\rho \to -\infty} y = \min \{x_1, \ldots, x_n\}$$

Answer: Let us assume that $\alpha_i = \alpha_j$. Then the CES production function has the form $y = (x_1^\rho + x_2^\rho)^{1/\rho}$. Let us suppose that $x_1 = \min(\sum x_i)$ and $\rho < 0$. We want to show that $x_1 = \lim_{\rho \to -\infty} (\sum x_i^\rho)^{1/\rho}$. Since all commodities x_i are required to be nonnegative, we can establish $x_1^\rho \leq \sum x_i^\rho$. Thus, $x_1 \geq (\sum x_i^\rho)^{1/\rho}$. On the other hand, $\sum x_i^\rho \leq n * x_1^\rho$. Hence $(\sum x_i^\rho)^{1/\rho} \geq n^{1/\rho} * x_1$. Letting $\rho \to -\infty$, we obtain $\lim_{\rho \to -\infty} (\sum x_i^\rho)^{1/\rho} = x_1$, because $\lim_{\rho \to -\infty} n^{1/\rho} * x_1 = x_1$.

3.2 Cost

3.19 What restrictions must there be on the parameters of the Cobb-Douglas form in Example 3.4 in order that it be a legitimate cost function?

Answer: The parameters A, w_1, w_2 and y are required to be larger than zero. A cost function is required to be increasing in input prices. Therefore, the exponents α and β must be larger zero. To fulfill the property of homogeneity of degree one in input prices, the exponents have to add up to one. To secure concavity in input prices, the second order partials should be less than zero. Thus, each of the exponents can not be larger one.

3.24 Calculate the cost function and conditional input demands for the Leontief production function in Exercise 3.8.

Answer This problem is identical to the expenditure function and compensated demand functions in the case of perfect complements in consumer theory.

Because the production is a min-function, set the inside terms equal to find the optimal relationship between x_1 and x_2. In other words, $\alpha x_1 = \beta x_2$. For a given level of output y, we must have $y = \alpha x_1 = \beta x_2$. Rearrange this expression to derive the conditional input demands:

$$x_1(\mathbf{w}, y) = \frac{y}{\alpha} \qquad x_2(\mathbf{w}, y) = \frac{y}{\beta}.$$

The cost function is obtained by substituting the two conditional demands into the definition of cost:

$$c(\mathbf{w}, y) = w_1 x_1(\mathbf{w}, y) + w_2 x_2(\mathbf{w}, y) = \frac{w_1 y}{\alpha} + \frac{w_2 y}{\beta}.$$

3.27 In Fig. 3.85, the cost functions of firms A and B are graphed against the input price w_1 for fixed values of w_2 and y.
(a) At wage rate w_1^0, which firm uses more of input 1? At w_1'? Explain?
Answer: Input demand can be obtained by using Shephard's lemma, represented by the slope of the cost function. Therefore, at w_1^0 firm B demands more of factor 1 and at wage rate w_1' firm A has a higher demand of that input.
(b) Which firm's production function has the higher elasticity of substitution? Explain.
Answer: The first-order conditions for cost minimisation imply that the marginal rate of technical substitution between input i and j equals the ratio of factor prices w_i/w_j. In the two input case, we can re-write the original definition of the elasticity of substitution as

$$\sigma = \frac{\mathrm{d}\ln(x_2/x_1)}{\mathrm{d}\ln(f_1/f_2)} = \frac{\mathrm{d}\ln(x_2/x_1)}{\mathrm{d}\ln(w_1/w_2)} = \frac{\hat{x}_2 - \hat{x}_1}{\hat{w}_1 - \hat{w}_2},$$

where the circumflex denotes percentage change in input levels and input prices, respectively. Because $\hat{w}_2 = 0$, the denominator reduces to $\hat{w}_1$, which is assumed to be the same for both firms. In (a) we established that input demand at w_1^0 is larger for firm B compared to firm A. It follows that the numerator will be larger for B and, subsequently, firm A's production function shows the higher elasticity of substitution at w_1^0.

3.29 The output elasticity of demand for input x_i is defined as

$$\epsilon_{iy}(w, y) \equiv \frac{\partial x_i(\mathbf{w}, y)}{\partial y} \frac{y}{x_i(\mathbf{w}, y)}.$$

(a) Show that $\epsilon_{iy}(\mathbf{w}, y) = \phi(y)\epsilon_{iy}(\mathbf{w}, 1)$ when the production function is homothetic. Given a homothetic production function, the cost function can be written as $c(\mathbf{w}, y) = \phi(y)c(\mathbf{w}, 1)$. Shephard's lemma states that the first order partial derivative with respect to the price of input i gives demand of x_i and to obtain the elasticity we need to take take the second-order cross-partial derivative of the cost function with respect to output. However, by Young's theorem it is known that the order of differentiation does not matter. Therefore, the following partial derivatives should be equal:

$$\frac{\partial^2 c(\mathbf{w}, y)}{\partial w_i \partial y} = \frac{\partial mc}{\partial w_i} = \frac{\partial x_i}{\partial y}.$$

Putting everything together gives:

$$\epsilon_{iy}(w, y) = \frac{\partial^2 c}{\partial y \partial w_i} \frac{y}{\partial c} \partial w_i = \frac{\partial \phi(y)}{\partial y} x_i(\mathbf{w}, 1) \frac{y}{\phi(y) x_i(\mathbf{w}, 1)} = \frac{1}{\phi'(y)} \epsilon_{iy}(\mathbf{w}, 1).$$

Unfortunately, this is not the result we should get.

(b) Show that $\epsilon_{iy} = 1$, for $i = 1, \ldots, n$, when the production function has constant returns to scale.
Answer For any production function with constant returns to scale, the conditional input demand x_i is linear in output level y (see Theorem 3.4). More formally, the conditional input demand of a production function homogeneous of degree $\alpha > 0$ can be written as $x_i(\mathbf{w}, y) = y^{1/\alpha} x_i(\mathbf{w}, 1)$. By definition, a constant returns to scale technology requires a production function homogeneous of degree 1. Therefore, the conditional input demand reduces to $x_i(\mathbf{w}, y) = y x_i(\mathbf{w}, 1)$. Calculating the output elasticity of demand for input x_i results in:

$$\epsilon_{iy}(w, y) \equiv \frac{\partial x_i(\mathbf{w}, y)}{\partial y} \frac{y}{x_i(\mathbf{w}, y)} = x_i(\mathbf{w}, 1) \frac{y}{y x_i(\mathbf{w}, 1)} = 1.$$

3.33 Calculate the cost function and the conditional input demands for the linear production function $y = \sum_{i=1}^{n} \alpha_i x_i$.
Answer Because the production function is linear, the inputs can be substituted for another. The most efficient input (i.e. input with the greatest marginal product/ price) will be used and the other inputs will not be used.

$$x_i(\mathbf{w}, y) = \begin{cases} \frac{y}{\alpha_i} & \text{if } \frac{\alpha_i}{w_i} > \frac{\alpha_j}{w_j} \forall j \neq i,\ j \in \{1, \ldots, n\} \\ 0 & \text{if } \frac{\alpha_i}{w_i} < \frac{\alpha_j}{w_j} \text{ for at least one } j \neq i,\ j \in \{1, \ldots, n\}. \end{cases}$$

The cost function is then $c(\mathbf{w}, y) = \frac{w_i y}{\alpha_i}$, where i is the input where

$$\frac{\alpha_i}{w_i} > \frac{\alpha_j}{w_j}\ \forall j \neq i,\ j \in 1, \ldots, n.$$

3.3 Duality in production

Additional exercise (Varian (1992) 1.6**)** For the following "cost functions" indicate which if any of properties of the cost function fails; e.g. homogeneity, concavity, monotonicity, or continuity. Where possible derive a production function.

(a) $c(\mathbf{w}, y) = y^{1/2}(w_1 w_2)^{3/4}$
Homogeneity: $c(t\mathbf{w}, y) = y^{1/2}(tw_1 tw_2)^{3/4} = t^{3/2}\left(y^{1/2}(w_1 w_2)^{3/4}\right)$ The function is not homogeneous of degree one.
Monotonicity:

$$\frac{\partial c(\mathbf{w}, y)}{\partial w_1} = 3/4 y^{1/2} w_1^{-1/4} w_2^{3/4} > 0 \frac{\partial c(\mathbf{w}, y)}{\partial w_2} = 3/4 y^{1/2} w_1^{3/4} w_2^{-1/4} > 0$$

The function is monotonically increasing in input prices.

Concavity:

$$\begin{aligned}
\mathbf{H} &= \begin{bmatrix} -\frac{3}{16}y^{1/2}w_1^{-5/4}w_2^{3/4} & \frac{9}{16}y^{1/2}w_1^{-1/4}w_2^{-1/4} \\ \frac{9}{16}y^{1/2}w^{-}1/4_1 w_2^{-1/4} & -\frac{3}{16}y^{1/2}w_1^{3/4}w_2^{-5/4} \end{bmatrix} \\
|H_1| &< 0 \\
|H_2| &= -\frac{72}{256}\frac{y}{\sqrt{w_1 w_2}} < 0
\end{aligned}$$

The function is not concave in input prices.
Continuity: Yes

(b) $c(\mathbf{w}, y) = \sqrt{y}(2w_1^{1/2}w_2^{1/2})$
The function satisfies all properties. The underlying technology is represented by $y = x_1 x_2$.

(c) $c(\mathbf{w}, y) = y(w_1 + \sqrt{w_1 w_2} + w_2)$
The function satisfies all properties. The underlying technology is represented by $y = 2/3\left((x_1 + x_2) + \sqrt{x_1^2 - x_1 x_2 + x_2^2}\right)$.

(d) $c(\mathbf{w}, y) = y(w_1 e^{-w_1} + w_2)$
The function is not homogeneous of degree one. Using Euler's Theorem we get the result $\sum \frac{\partial c}{\partial w_i} w_i = y(w_1 e^{-w_1} + w_1^2 e^{-w_1} + w_2)$ what is clearly not equal to the original cost function. Alternatively, it becomes clear from the expression $c(t\mathbf{w}, y) = ty(w_1 e^{-tw_1} + w_2)$. The function is not monotonically increasing in input prices as the first partial derivative with respect to w_1 is only positive for prices less than one: $\partial c/\partial w_1 = ye^{-w_1}(1-w_1)$. Furthermore the function is only concave for prices $w_1 < 2$, what can be seen from the first determinant of the Hessian matrix: $|H_1| = y(w_1 - 2)e^{-w_1}$.

(e) $c(\mathbf{w}, y) = y(w_1 - \sqrt{w_1 w_2} + w_2)$
Monotonicity of the cost function holds only for a narrow set of input prices with the characteristics $1/4w_2 < w_1 < 4_4$. The conclusion can be derived from the first partial derivatives and a combination of the two inequalities.

$$\begin{aligned}
\frac{\partial c}{\partial w_1} &= y\left(1 - \frac{1}{2}\sqrt{\frac{w_2}{w_1}}\right) \text{ positive for } 1 > \frac{1}{2}\sqrt{\frac{w_2}{w_1}} \\
\frac{\partial c}{\partial w_2} &= y\left(1 - \frac{1}{2}\sqrt{\frac{w_1}{w_2}}\right) \text{ positive for } 1 > \frac{1}{2}\sqrt{\frac{w_1}{w_2}} \text{ or } 2 > \sqrt{\frac{w_2}{w_1}}
\end{aligned}$$

The function is not concave as the first partial derivatives with respect to both input prices are negative and the second-order partial derivatives are positive. The determinants of the Hessian matrix are $|H_1| > 0$ and $|H_2| = 0$. Thus, the function is convex.

(f) $c(\mathbf{w}, y) = (y + 1/y)\sqrt{w_1 w_2}$
The function satisfies all properties, except continuity in $y = 0$.

3.40 We have seen that every Cobb-Douglas production function, $y = Ax_1^{\alpha} x_2^{1-\alpha}$, gives rise to a Cobb-Douglas cost function, $c(\mathbf{w}, y) = yAw_1^{\alpha} w_2^{1-\alpha}$, and every CES production function, $y = A\left(x_1^{\rho} + x_2^{\rho}\right)^{1/\rho}$, gives rise to a CES cost function, $c(\mathbf{w}, y) = yA\left(x_1^{r} + x_2^{r}\right)^{1/r}$. For each pair of functions, show that the converse is also true. That is, starting with the respective cost functions, "work backward" to the underlying production function and show that it is of the indicated form. Justify your approach.
Answer: Using Shephard's lemma we can derive the conditional input demand functions. The first step to solve this exercise for a Cobb-Douglas cost function is to derive Shephard's lemma and to rearrange all input demands in such a way to isolate the ratio of input prices on one side, i.e. left-hand side of the expression. On the right-hand side we have the quantity of input(s) and output. Second, equalise the two expressions and solve for y. The result will be the corresponding production function.

$$x_1 = \frac{\partial c(\mathbf{w}, y)}{\partial w_1} = \alpha y A\left(\frac{w_2}{w_1}\right)^{1-\alpha} \qquad \frac{w_2}{w_1} = \left(\frac{x_1}{A\alpha y}\right)^{1/(1-\alpha)}$$
$$x_2 = \frac{\partial c(\mathbf{w}, y)}{\partial w_2} = (1-\alpha) y A\left(\frac{w_2}{w_1}\right)^{-\alpha} \qquad \frac{w_2}{w_1} = \left(\frac{x_2}{A(1-\alpha)y}\right)^{-1/\alpha}$$
$$\frac{(A\alpha y)^{\alpha}}{x_1^{\alpha}} = \frac{x_2^{1-\alpha}}{(A(1-\alpha)y)^{1-\alpha}}$$

$$y = \frac{x_1^{\alpha} x_2^{1-\alpha}}{\alpha^{\alpha}(1-\alpha)^{1-\alpha}} = a x_1^{\alpha} x_2^{1-\alpha} \text{ where } a = \left(A\alpha^{\alpha}(1-\alpha)^{1-\alpha}\right)^{-1}$$

For the CES cost function a short-cut is used: Derive the conditional input demand functions and substitute them into the production function.

$$x_1 = \frac{\partial c(\mathbf{w}, y)}{\partial w_1} = yAw_1^{r-1}\left(w_1^r + w_2^r\right)^{\frac{1}{r}-1}$$
$$x_2 = \frac{\partial c(\mathbf{w}, y)}{\partial w_2} = yAw_2^{r-1}\left(w_1^r + w_2^r\right)^{\frac{1}{r}-1}$$
$$y = \left[(Ay)^{\rho}\frac{w_1^{-r} + w_2^{-r}}{w_1^{-r} + w_2^{-r}}\right]^{1/\rho} = Ay$$

3.4 The competitive firm

Additional exercise (Varian (1992) 1.21**)** Given the following production function $y = 100x_1^{1/2} x_2^{1/4}$.

(a) Find $c(w_1, w_2, y)$.
Answer: Starting from the equality of MRTS and ratio of factor prices, we get

$w_1/w_2 = 2x_2/x_1$. Solving for one of the inputs, substituting back in the production function and rearranging, we derive the **conditional** input demand functions:

$$x_1 = \left(\frac{y}{100}\right)^{4/3}\left(\frac{2w_2}{w_1}\right)^{1/3} \text{ and}$$
$$x_2 = \left(\frac{y}{100}\right)^{4/3}\left(\frac{2w_2}{w_1}\right)^{-2/3}.$$

Substituting the two functions in the definition of costs, the resulting cost function is:

$$\begin{aligned}c(w_1, w_2, y) &= \left(\frac{y}{100}\right)^{4/3} w_1^{2/3}(2w_2)^{1/3} + \left(\frac{y}{100}\right)^{4/3} w_2^{1/3}w_1^{2/3}2^{-2/3}\\ &= \left(2^{1/3} + 2^{-2/3}\right)\left(\frac{y}{100}\right)^{4/3} w_1^{2/3}w_2^{1/3}.\end{aligned}$$

(b) Find the effect of an increase in output on marginal cost, and verify that $\lambda =$ marginal cost.
Answer: Marginal costs are
$MC = \partial c/\partial y = \frac{1}{75}(y/100)^{1/3}w_1^{2/3}w_2^{1/3}\left(2^{1/3} + 2^{-2/3}\right)$. Marginal costs are increasing with output which is shown by
$\frac{\partial MC}{\partial y} = \frac{\partial^2 c}{\partial y^2} = \left(\frac{1}{150}\right)^2 (y/100)^{-2/3}w_1^{2/3}w_2^{1/3}\left(2^{1/3} + 2^{-2/3}\right)$. From the FOC of the Lagrangian we can derive that $\lambda^* = \frac{w_1 x_1^{1/2}}{50x_2^{1/4}} = \frac{w_2 x_2^{3/4}}{25x_1^{1/2}}$. Substituting the conditional input demand functions into one of those expressions gives

$$\begin{aligned}\lambda^* &= \frac{w_1}{50}\left((y/100)^{4/3}(2w_2/w_1)^{1/3}\right)^{1/2}\left((y/100)^{4/3}(2w_2/w_1)^{-2/3}\right)^{-1/4}\\ &= \frac{2^{1/3}}{50}\left(\frac{y}{100}\right)^{1/3}w_1^{2/3}w_2^{1/3}\end{aligned}$$

When you solve the ratios, this expression will be equal to the marginal cost function.

(c) Given $p =$ price of output, find $x_1(\mathbf{w}, p), x_2(\mathbf{w}, p)$ and $\pi(\mathbf{w}, p)$. Use Hotelling's lemma to derive the supply function $y(\mathbf{w}, p)$.
Answer: By maximising $\pi = py - c(\mathbf{w}, y)$ the first-order condition is

$$\begin{aligned}\frac{\partial \pi}{\partial y} =& p - \frac{1}{75}\left(\frac{y}{100}\right)^{1/3} w_1^{2/3}w_2^{1/3}(2^{1/3} + 2^{-2/3}) = 0\\ y =& 100\left(\frac{75}{2^{1/3} + 2^{-2/3}}\right)^3\left(\frac{p}{w_1^{2/3}w_2^{1/3}}\right)^3\end{aligned}$$

The first expression affirms the equality of price and marginal cost as the profit maximum for any competitive firm. The last expression gives already the profit maximising supply function. Furthermore, the two following unconditional demand functions emerge as solution of this optimisation problem:

$$x_1 = \left(\left(\frac{75p}{2^{1/3}+2^{-2/3}}\right)^3 w_1^{-2}w_2^{-1}\right)^{4/3}\left(\frac{2w_2}{w_1}\right)^{1/3} = \left(\frac{75}{2^{1/3}+2^{-2/3}}\right)^4 \frac{2^{1/3}p^4}{w_1^3 w_2}$$

$$x_2 = \left(\left(\frac{75p}{2^{1/3}+2^{-2/3}}\right)^3 w_1^{-2}w_2^{-1}\right)^{4/3}\left(\frac{w_1}{2w_2}\right)^{2/3} = \frac{75}{2^{1/3}+2^{-2/3}}\frac{p^4}{2^{2/3}w_1^2 w_2^2}.$$

The profit function is

$$\begin{aligned}
\pi &= \left(\frac{75}{2^{1/3}+2^{-2/3}}\right)^3 \frac{100p^4}{w_1^2 w_2} - \left(\frac{75p}{2^{1/3}+2^{-2/3}}\right)^4 \left(\frac{2^{1/3}w_1}{w_1^3 w_2} + \frac{w_2}{2^{2/3}w_1^2 w_2^2}\right) \\
&= \left(\frac{75}{2^{1/3}+2^{-2/3}}\right)^3 \frac{p^4}{w_1^2 w_2}(100-75) \\
&= 25\left(\frac{75}{2^{1/3}+2^{-2/3}}\right)^3 \frac{p^4}{w_1^2 w_2}.
\end{aligned}$$

Hotelling's lemma confirms the output supply function shown above.

(d) Derive the unconditional input demand functions from the conditional input demands.

Answer One, among several, way is to substitute the conditional input demands into the definition of cost to obtain the cost function. Calculating marginal cost and equalising with output price, gives, after re-arrangement, the output supply function. Substitution of the output supply function into the conditional input demands results in the unconditional input demand functions. Using the example at hand, and starting from the equality $\partial c/\partial y = p$ gives:

$$\begin{aligned}
p &= \frac{1}{75}\left(2^{1/3}+2^{-2/3}\right)\left(\frac{y}{100}\right)^{1/3} w_1^{2/3}w_2^{1/3} \\
y &= 75^3\left(2^{1/3}+2^{-2/3}\right)^{-3} p^3 w_1^{-2} w_2^{-1}\cdot 100 \\
x_1(\mathbf{w},p) &= \left(\frac{75}{2^{1/3}+2^{-2/3}}\right)^4 p^4 w_1^{-3} w_2^{-1}
\end{aligned}$$

(e) Verify that the production function is homothetic.

Answer: The cost function is a factor of a function of output and input prices. Similarly, the conditional input demand functions are products of a function of y and input prices. Therefore, the possibility to separate the two functions multiplicatively and following Theorem 3.4 shows that the production function has to be a homothetic function.

(f) Show that the profit function is convex.

Answer: In order to simply this step, I write the constant part of the profit function as $K = 25\left(\frac{75}{2^{1/3}+2^{-2/3}}\right)^3$ Calculating the second-order partial derivatives of the profit function with respect to all prices gives the following Hessian matrix. Be aware of the doubble sign change after each derivative with respect to the input price (check Theorem 3.8).

$$\mathbf{H} = \begin{bmatrix} \frac{12Kp^2}{w_1^2 w_2} & \frac{-8Kp^3}{w_1^3 w_2} & \frac{-4Kp^3}{w_1^2 w_2^2} \\ -\frac{-8Kp^3}{w_1^3 w_2} & -\frac{-6Kp^4}{w_1^4 w_2} & -\frac{-2Kp^4}{w_1^3 w_2^2} \\ -\frac{-4Kp^3}{w_1^2 w_2^2} & -\frac{-2Kp^4}{w_1^3 w_2^2} & -\frac{-2Kp^4}{w_1^2 w_2^3} \end{bmatrix}$$

The own supply effect is positive, the own demand effects are negative and all cross-price effects are symmmetric. Checking the determinants becomes quite tedious. Intuituively, it should become clear that they all have to be positive.

(g) Assume x_2 as a fixed factor in the short run and calculate short-run total cost, short-run marginal cost, short-run average cost and short-run profit function.
Short-run total cost are obtained by re-arranging the production function to get x_1 on the left-hand side and plugging in into the definition of cost $c(\mathbf{w}, y) = (y/100)^2 w_1/x_2^{1/2} + w_2 x_2$. The first-partial derivative gives the short-run marginal cost function $smc = \frac{1}{50}\frac{y}{100} w_1/x_2^{1/2}$. The short-run average costs are equal to $sac = \frac{y}{100^2} w_1/x_2^{1/2} + \frac{w_2 x_2}{y}$.

Final remark: Some answers might not be the most elegant ones from a mathematical perspective. Any comment and suggestion, also in case of obscurities, are highly welcome.

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